Overview of Fractions

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This presentation is intended for students and parents who have seen this material before but need to review. The focus here is on procedures. This is not the whole story! A conceptual foundation is also important, especially for parents presenting the material to their children for the first time. A conceptual overview will be presented separately.

Multiplication: Multiply straight across: Multiply the numerators to get the new numerator. Multiply the denominators to get the new denominator.

$$\overset{\rightarrow}{\underset{\rightarrow}{\frac{3}{4}\times\frac{1}{2}=\frac{3}{8}}}$$

Division: The first fraction is said to be "divided by" the second fraction. The fraction you are "dividing by" is called the "divisior." To divide, "invert" the divisor (turn it upside down), then multiply.

$$\frac{1}{2}\div\frac{2}{3}=? \quad \rightarrow \text{ change } \frac{2}{3} \text{ to } \frac{3}{2} \text{ and multiply: } \rightarrow \quad \frac{1}{2}\times\frac{3}{2}=\frac{3}{4}$$

(When a fraction is inverted we call the resulting fraction the "reciprocal" of the original number. Any whole number can be thought of as a fraction with 1 in the denominator. Since $2=\frac{2}{1}$, the reciprocal of 2 is $\frac{1}{2}$.)

Chain Multiplication and Division: There are times (especially in science classes in high school and beyond) when you need to multiply and divide a whole series of fractions and whole numbers. This is actually easy to do.

Think of making a shish kabob: a skewer loaded up with chunks of meat, tomatoes, onions, bell pepper, mushrooms, etc. and cooked on a grill. Think of one long fraction bar as a skewer. To multiply a fraction in the list, simply skewer it on. To divide by a fraction, flip it over and skewer it on. Any whole number can be thought of as a fraction with 1 in the denominator, so can treat it like any other fraction. Multiplying by a whole number puts the whole number into the numerator. Dividing by a whole number puts the whole number into the denominator. (Once everything is in place we can ignore the 1's.)

$$\frac{2}{3}\times\frac{4}{5}\div\frac{7}{8}\times 5\div 7 \quad\rightarrow\quad \frac{2\times 4\times 8\times 5\times 1}{3\times 5\times 7\times 1\times 7} \quad\rightarrow\quad \frac{2\times 4\times 8\times 5}{3\times 5\times 7\times 7}$$

(We'll finish this problem later when we talk about cancellation.)

Equivalent fractions: If the top *and* bottom of any fraction are multiplied by the same number, the resulting fraction *looks* different, but it is equal to the original fraction. Finding equivalent fractions is used to reduce fractions, to find common denominators, and for many other things later on in math.

$$\frac{3}{4}=\frac{3\times 2}{4\times 2}=\frac{6}{8};\quad \frac{3}{4}=\frac{3\times 3}{4\times 3}=\frac{9}{12};\quad \frac{3}{4}=\frac{3\times 4}{4\times 4}=\frac{12}{16};\ \ \ldots\ \text{etc.}$$

$\frac{3}{4}$, $\frac{6}{8}$, $\frac{9}{12}$, and $\frac{12}{16}$ are all equivalent fractions. (Try converting all of these to decimals on a calculator by dividing $3\div 4$, $6\div 8$, etc. What do you notice?)

Reducing Fractions: This process is used to express fractions in simpler form. Break down the numerator and denominator into their factors (e.g. 6 breaks down into 3×2). If any factors appear in both the numerator and denominator, cancel them out.

$$\frac{12}{18} = \frac{3 \times 4}{2 \times 9} = \frac{3 \times 2 \times 2}{2 \times 3 \times 3} = \frac{2}{3} \text{ (a pair of 2's and a pair of 3's cancel out)}$$

Cancellation: Consider the following multiplication problem.

$$\rightarrow$$

$$\frac{2}{3} \times \frac{3}{4} = \frac{6}{12}$$

$$\rightarrow$$

The answer needs to be reduced. To reduce it we would spread out the numerator and denominator into their factors and cross out the ones that appear in both numerator and denominator.

$$\frac{2}{3} \times \frac{3}{4} = \frac{6}{12} = \frac{2 \times 3}{3 \times 2 \times 2} = \frac{1}{2}$$

Notice that factoring the result *undoes the work we just did* in multiplying the original numbers. We would have been better off to *reduce first and multiply only the factors that survive*. Let's do the same problem again, reducing before multiplying:

$$\frac{2}{3} \times \frac{3}{4} = \frac{2 \times 3}{3 \times 2 \times 2} = \frac{1}{2}$$

A pair of 2's and a pair of 3's cancel out, leaving only a 2 in the denominator.

Notice that everything in the numerator cancels out. When this happens, what is left behind is not a 0, but rather a 1. That is because 1 is always a "hidden" factor in every number. It's not written down unless it is needed. In this problem the numerator 2×3 could be thought of as $1 \times 2 \times 3$, so when the 2 and 3 both cancel, what is left is the 1.

Let's finish the problem from the section on *Chain Multiplication and Division*:

$$\frac{2}{3} \times \frac{4}{5} \div \frac{7}{8} \times 5 \div 7 \quad \rightarrow \quad \frac{2 \times 4 \times 8 \times 5}{3 \times 5 \times 7 \times 7} = \frac{2 \times 4 \times 8}{3 \times 7 \times 7} = \frac{64}{147}$$

Set up the problem as before, cancel the factors that appear in both numerator and denominator (in this case only the 5's), then multiply the surviving factors.

[Warning: Do not cancel in division problems until you have converted them to multiplication problems!!]

Addition (when the denominators are the same): Add the numerators. Keep the same denominator.

$$\text{Addition:} \qquad \frac{3}{7}+\frac{1}{7}=\frac{3+1}{7}=\frac{4}{7}$$

Subtraction (when the denominators are the same): Subtract the numerators. Keep the same denominator.

$$\text{Subtraction:} \quad \frac{3}{7}-\frac{1}{7}=\frac{3-1}{7}=\frac{2}{7}$$

Common Denominators: Since we can change the form of a fraction by multiplying the top and bottom by the same thing, we can force any two fractions to have the same denominator. The trick is to give both denominators all the same factors.

$$\frac{2}{3} \text{ and } \frac{3}{5} \quad \rightarrow \quad \frac{2\times 5}{3\times 5} \text{ and } \frac{3\times 3}{5\times 3} \quad \rightarrow \quad \frac{10}{15} \text{ and } \frac{9}{15}$$

Note that if the denominators we want to match are 3 and 5, we can multiply 3 by 5 and 5 by 3 to convert both fractions to 15ths. (We actually didn't *change* the fractions in the process, because we multiplied both top and bottom of each fraction by the same number in each case. That's what keeps it all "legal.")

Least Common Denominators: If two denominators have some factors in common, simply multiplying the denominators together will give you needlessly big numbers that have to be reduced later. Instead, spread out and compare the factors of both denominators and multiply each denominator by the factors it lacks.

$$\begin{array}{ccc} \dfrac{1}{12} & \text{and} & \dfrac{1}{16} \\ 3\times 2\times 2 & & 2\times 2\times 2\times 2 \end{array}$$

Notice that both denominators contain 2×2 (marked in blue), but one has an extra 3 and the other has two more extra 2's. If the first fraction is multiplied (top and bottom) by 4 $(=2\times 2)$ and the second fraction is multiplied (top and bottom) by 3, all the factors in both denominators will match.

$$\begin{array}{ccccccc} \dfrac{1}{12} & \text{and} & \dfrac{1}{16} & = & \dfrac{4\times 1}{4\times 12}=\dfrac{4}{48} & \text{and} & \dfrac{3\times 1}{3\times 16}=\dfrac{3}{48} \\ 3\times 2\times 2 & & 2\times 2\times 2\times 2 & & & & \end{array}$$

The little bit of extra work breaking the denominators into factors at this stage is easier than multiplying 12 by 16 and 16 by 12 (= 192), because you would then have to turn right around and reduce the resulting fractions which would by then involve much larger numbers.

Addition or Subtraction when the denominators are NOT the same: Convert the fractions to equivalent fractions having the same denominator, then add or subtract the numerators as before.

$$\begin{array}{cccccccc} \dfrac{1}{12} & + & \dfrac{1}{16} & = & \dfrac{4\times 1}{4\times 12}+\dfrac{3\times 1}{3\times 16} & = & \dfrac{4}{48}+\dfrac{3}{48} & = & \dfrac{7}{48} \\ 3\times 2\times 2 & & 2\times 2\times 2\times 2 & & & & & & \end{array}$$

Converting Mixed Numbers to "Improper" Fractions: Mixed numbers (or mixed *numerals*, as they are sometimes called) represent the sum of a whole number and a fraction.

$$2\frac{3}{4} = 2 + \frac{3}{4}$$

You can convert the mixed number to a fraction by thinking of the whole number as a fraction with 1 in the denominator, then adding:

$$2\frac{3}{4} = \frac{2}{1} + \frac{3}{4} = \frac{2\times 4}{1\times 4} + \frac{3}{4} = \frac{(2\times 4)+3}{4} = \frac{11}{4}$$

We can cut out all the intermediate steps to get a simple rule:

- The new denominator will always be the same as the denominator of the fractional part (4, in this example).
- To get the new numerator, multiply the denominator by the whole number and add the old numerator (4 times 2 is 8, plus 3 is 11).

Notice that the result, $\frac{11}{4}$, is a fraction where the numerator is larger than the denominator. If something is cut into fourths, 4 fourths would be the whole thing. 11 fourths would make a lot more than 1 whole thing. Improper fractions are fractions where the numerator is larger than the denominator-- fractions whose value is greater than 1. The terminology is unfortunate. There is nothing *wrong* (or *improper*) about improper fractions.

Converting "Improper" Fractions to Mixed Numbers (and why you might not always want to do this): When you want to use an improper fraction and when you want to use a mixed number depends on what you're trying to do. Consider $2\frac{1}{2}$, which equals $\frac{5}{2}$. Mixed numbers are messy to compute with. If you are planning to use the number in further computations, $\frac{5}{2}$ is more convenient. If you want to measure off a length with a ruler, $2\frac{1}{2}$ lets you find the whole number of inches, then worry about the last fraction of an inch separately, so the mixed number is more convenient.

To convert an improper fraction into a mixed number, divide the numerator by the denominator and express the remainder as a fraction.

$$2\overline{)5}\ \ 2\tfrac{1}{2}$$

Adding Mixed Numbers
To add mixed numbers:

- Add the fractional parts
- If the resulting fractional part is greater than one, convert it to a whole number plus a possible fractional remainder.
- Add the whole numbers, together with whatever is carried over from the sum of the fractions.

Example: Add $5\frac{2}{3}+2\frac{3}{4}$.

Note that the problem can be rewritten, $(5+2)+\left(\frac{2}{3}+\frac{3}{4}\right)$

Add the fractional parts: $\frac{2\cdot 4}{3\cdot 4}+\frac{3\cdot 3}{4\cdot 3}=\frac{8+9}{12}=\frac{17}{12}=1\frac{5}{12}$

Now add the whole numbers, together with the whole number part of the fraction result:

$$5+2+1\frac{5}{12}=(5+2+1)+\frac{5}{12}=8\frac{5}{12}$$

This process is often taught with the numbers written vertically:

$$\begin{array}{l} 5\frac{2}{3}\rightarrow\frac{8}{12} \\ +2\frac{3}{4}\rightarrow\frac{9}{12} \\ \hline 7\ \ +\ \frac{17}{12}=8\frac{5}{12} \end{array}$$

Subtracting Mixed Numbers
Subtracting mixed numbers proceeds exactly like addition (subtract the fractional parts; subtract the whole number parts), with one possible complication. There won't be any carrying, but you may have to borrow if you are subtracting a larger fraction from a smaller one. Consider the previous example transformed into a subtraction problem:

$$\begin{array}{l} 5\frac{2}{3}\rightarrow\frac{8}{12} \\ -2\frac{3}{4}\rightarrow\frac{9}{12} \\ \hline \end{array}$$

Note that we are subtracting $\frac{9}{12}$ from $\frac{8}{12}$. The procedure here is to borrow 1 from the 5, rewrite it as $\frac{12}{12}$, and combine this with the $\frac{8}{12}$ for a total of $\frac{20}{12}$. Then proceed with the problem:

$$\begin{array}{rr} 5\frac{2}{3} \rightarrow \frac{8}{12} & 4\frac{8+12}{12} \\ \underline{-2\frac{3}{4} \rightarrow \frac{9}{12}} & \underline{2\frac{9}{12}\quad} \\ & 2\frac{11}{12} \end{array}$$